

Department of Mathematics

MATH 222 - Discrete and Combinatorial  
Mathematics

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## Answers and Hints to Practice Questions

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Kieka Mynhardt's notes, assignments, and tests for Math 222

Introduction to Combinatorics and Graph Theory - Custom Edition for the University of Victoria

Discrete Mathematics: Study Guide for MAT212-S - Dr. Kieka Myndardt

Discrete Mathematics - Norman L. Biggs

Applied Combinatorics, fourth edition - Alan Tucker

Discrete Mathematics, An Introduction to Mathematical Reasoning - Susanna S. Epp

Discrete Mathematics with Combinatorics - James A. Anderson

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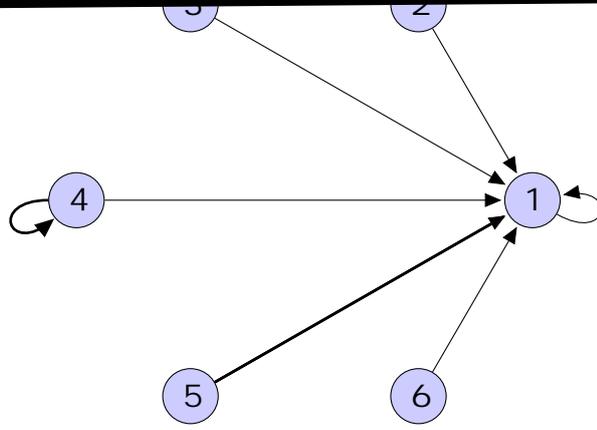


5. Hint: Show  $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$

## 1.2 Relations and Graphs

### Solutions:

1. (a) Symmetric.  
  
(b) Reflexive, symmetric.  
  
(c) Reflexive, transitive, antisymmetric.  
  
(d) Reflexive, transitive, antisymmetric.
  
2. (a) The equivalence class of 11 is  $[11]$  since 11 is prime. The other positive integers in  $[11]$  are the positive integers whose largest prime divisor is 11. The number of equivalence classes is equal to the cardinality of all the prime numbers, which is infinite.  
  
(b) The equivalence class of  $(2; 5)$  is  $[29]$ . This equivalence class includes all ordered pairs of real numbers such that  $x_1^2 + x_2^2 = 29$ .  
  
There are infinitely many equivalence classes, one for each positive real number that can be written as the sum of the squares of two real numbers.



$$4. R = f(1; 1); (3; 3); (6; 6); (1; 4). R$$

## 2 Graph Theory

(b)  $m \cdot n$  edges.

3. 16 edges.

4. (a) Does not exist.

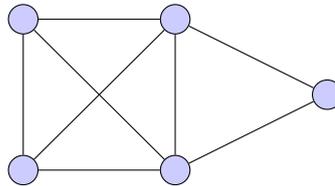
(b) Does not exist.

(c) Does not exist.

(d) One possible graph:



7. One possible graph:



8. Three.

9. Four.

10. Hint: Euler's Theorem.

11. No.

Hint: Cases.

12. Yes. This answer does not change if we require non-empty edges sets.

13. d 2 f1; 2 3 5g;

14. (a)



(b) Does not exist.

(c) Does not exist.

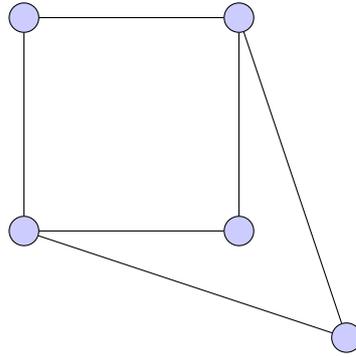
15.  $2^{\frac{n(n-1)}{2}}$ .

16. 23 vertices.

17. Hint: Euler's Theorem.

18.  $n - 1$ .

19. One possible graph:



20. Impossible.

21. Yes.

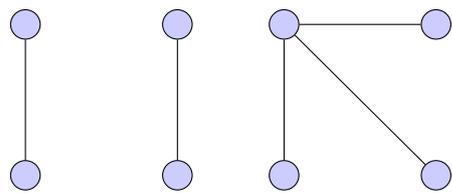
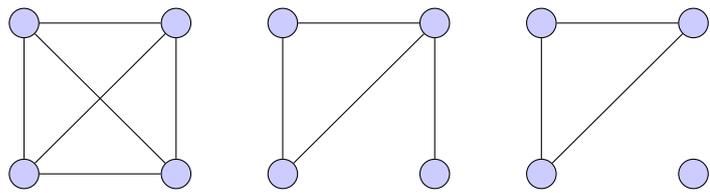
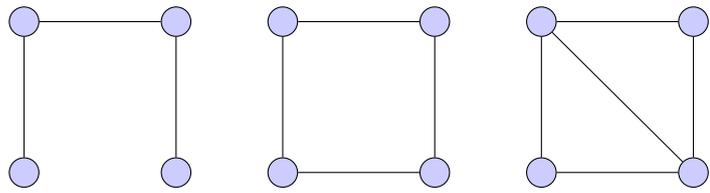
22. Hint: Consider all possible degrees of an arbitrary vertex.

23. Hint: Proof by contradiction.

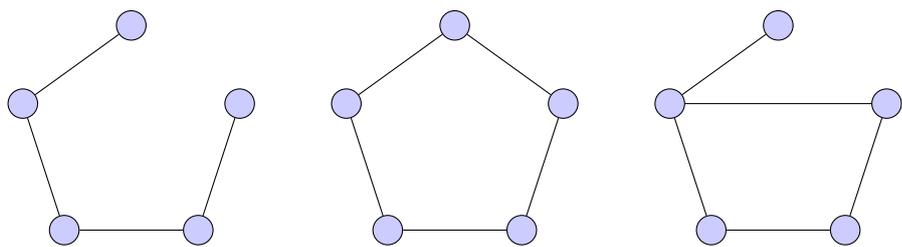
## 2.3 Isomorphisms

Solutions:

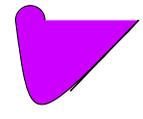
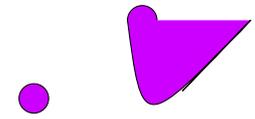
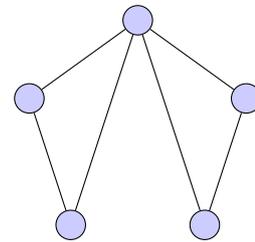
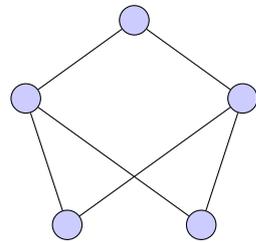
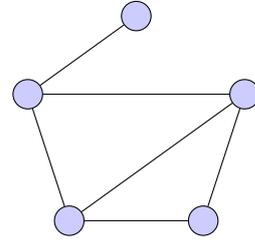
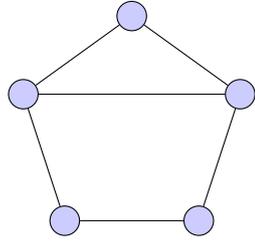
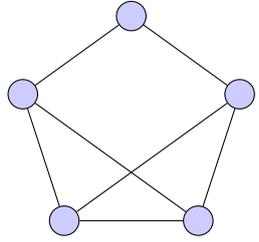
1. Two isomorphic graphs have the same structure.
2. The converse of this statement: If two graphs  $G_1$  and  $G_2$  have the same number of vertices, same number of edges, and the same degree sequence, then they



(c)







f ! s  
g ! t  
h ! u

7.

110

111

101

010

000

011

001

(d) Isomorphic.

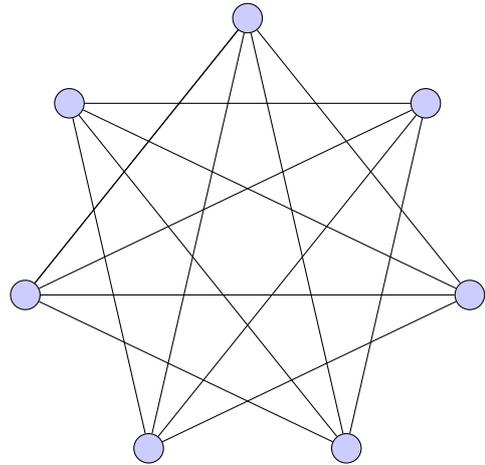
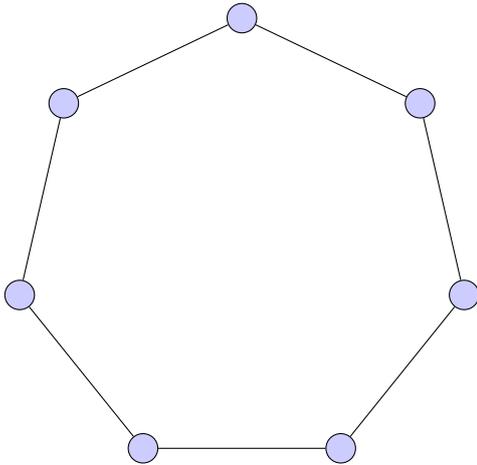
(e) Non-isomorphic.

10. Hint: There are exactly ten self-complementary graphs of order 8.

## 2.4 Eulerian Circuits

### Solutions:

1. An Eulerian circuit is a trail that uses every edge exactly once and ends where it began. A Eulerian trail is a trail that goes through every edge, but does not necessarily end where it began.
  
2. (a) No Eulerian circuit nor Eulerian trail.  
(b) No Eulerian circuit exists but there exists an Eulerian trail.  
(c) Both Eulerian circuit and Eulerian trail.  
(d) No Eulerian circuit nor Eulerian trail.  
(e) No Eulerian circuit nor Eulerian trail.  
(f) No Eulerian circuit nor Eulerian trail.
  
3. Not Eulerian.
  
4. Hint: Recall the Königsberg Bridge Problem: The city of Königsberg, Prussia, was set on both sides of a river and included two large islands, all connected by seven bridges. Is there a way to walk through the city crossing every bridge exactly once while finishing where you started?
  
5. Hint: By definition, every path is also a walk.
  
6. One possible graph:



7. (a) False.  
 (b) True.  
 (c) False.  
 (d) True.

8.  $m, n$  both non-zero, even integers.

9. No.

10. (a)  $n$  odd.

(b) For all odd values of  $n$  there will be a closed Eulerian trail (i.e. an Eulerian circuit). The only open Eulerian trail occurs when  $n = 2$ .

11. True.

Hint: Consider an arbitrary circuit.

12. Hint: Show the three properties of an equivalence relation hold.

13. Hint: Consider two arbitrary vertices in opposite components of a disconnected  $G$ .

14. Hint: Proof by contradiction.

15.  $|E(G)| = n - 1$ .

16. Hint: In any circuit, there exist two trails between any given pair of vertices.

## 2.5 Hamiltonian Cycles

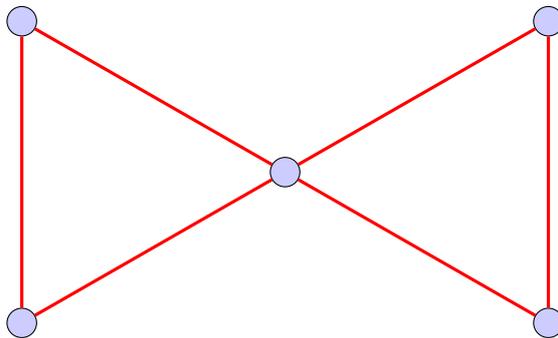
### Solutions:

1. (a) In a Eulerian circuit it is possible to pass through some vertices multiple times while that is not possible in a Hamiltonian cycle. Also, a Hamiltonian cycle may not visit every edge while that is a requirement of a Eulerian circuit.

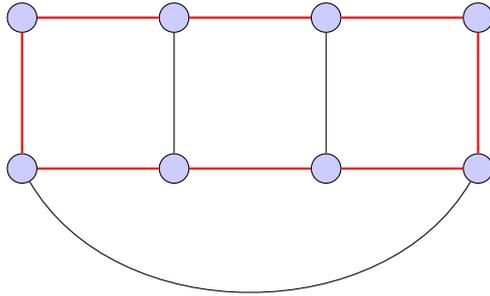
(b) A Eulerian trail may visit the same vertex multiple times while a Hamiltonian path will not. A Hamiltonian path may not visit every edge in the graph, while that is a requirement of Eulerian trail.

2. (a) Not Hamiltonian.

(b) The Hamiltonian cycle is highlighted in red.



(c) One such cycle is highlighted in red.

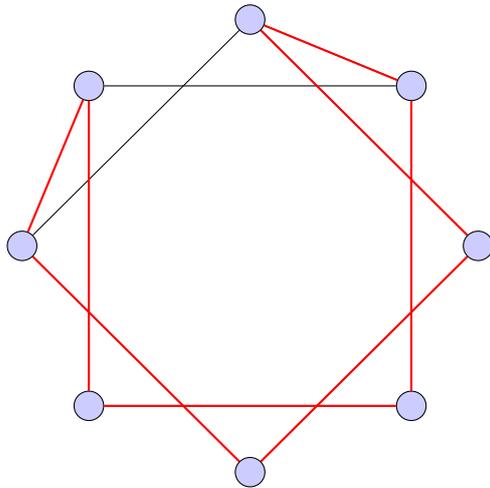


(d) Not Hamiltonian.

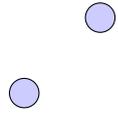
(e) Not Hamiltonian.

(f) Not Hamiltonian.

(g) Yes, a Hamiltonian cycle is highlighted in red:



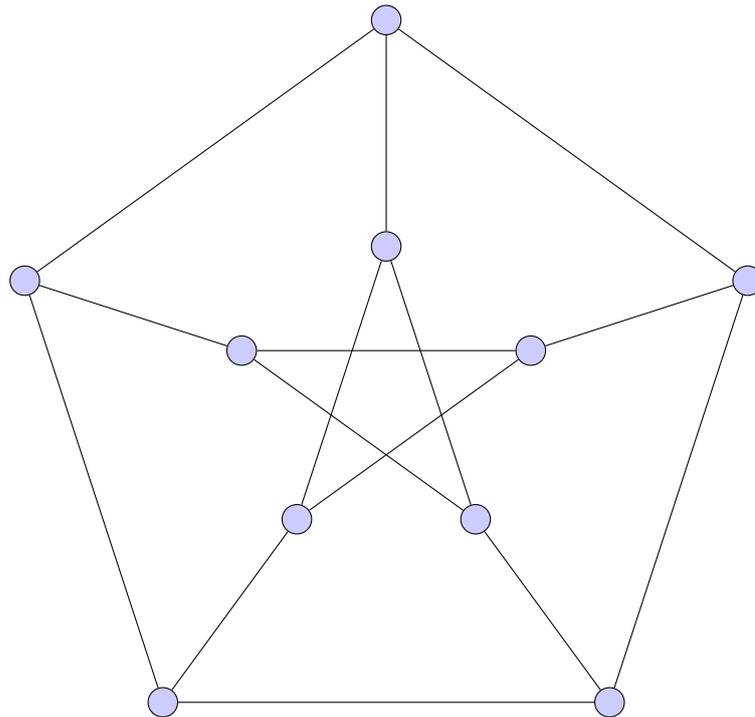
3. The following graph is a sufficient counterexample:



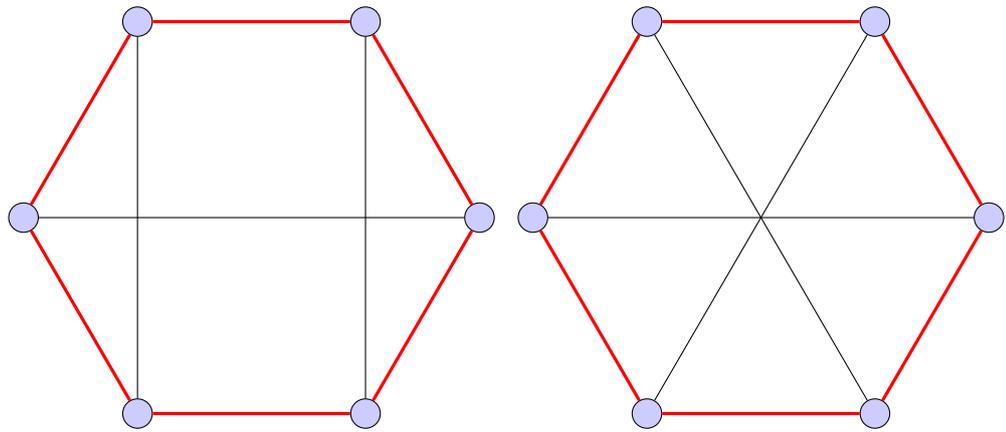
10. Hint: Consider an arbitrary cycle.

11. Hint: Proof by contradiction.

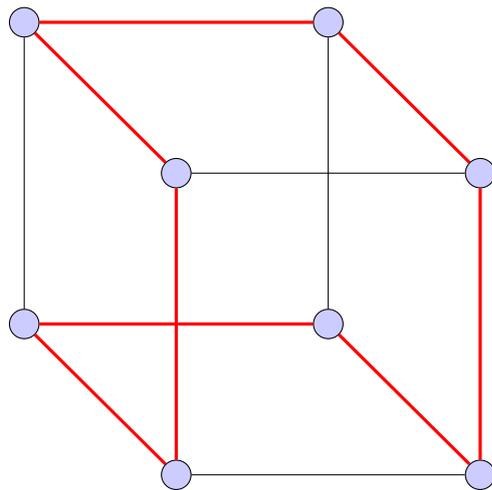
12. The Petersen graph is such a graph:



13. Here are the only such graphs, with Hamiltonian cycles highlighted in red.



14. Highlighted below is one of several Hamiltonian cycles in red:



## 2.6 Trees and Their Properties

### Solutions:

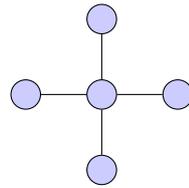
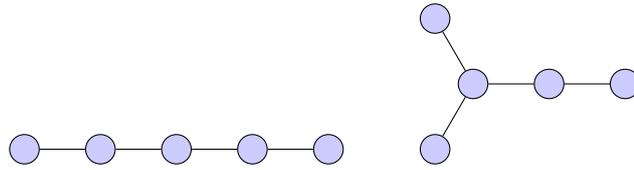
1. In nitely many possible degree sequences exist. If you can draw a tree with your degree sequence then it is correct.

2. The following are equivalent:

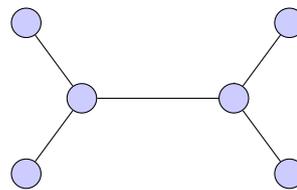
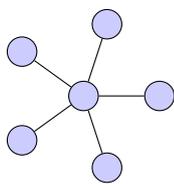
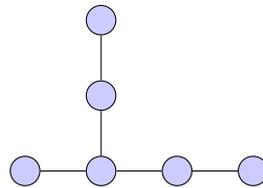
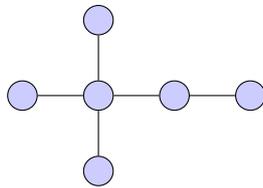
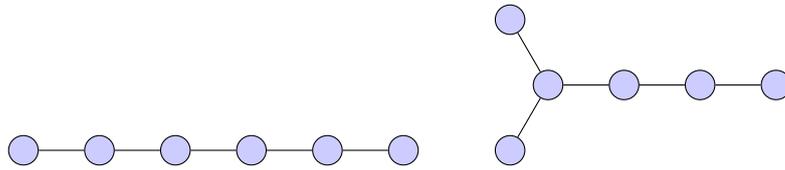
- i)  $G$  is a tree.
- ii)  $G$  is a connected acyclic graph.
- iii)  $G$  is a connected graph with  $n-1$  edges.
- iv)  $G$  is an acyclic graph with  $n-1$  edges.

3.1 ed[(1)-3o6(an)-32our tree.

(b)



(c)



5. Yes.

6. False.

7. Nine vertices.

8.  $x = 5$ .

9. Hint: Induction on the number of vertices.

10. 102 vertices.

11. There are two vertices of degree 5.

12. Hint: Use the alternate definitions of a tree.

The components of this new graph are trees.

13. Hint: Consider a property shared by trees and bipartite graphs.

14. Hint: Euler's Theorem.

15. A tree,  $T$ , is a complete bipartite graph if and only if  $K_{1;n}$  for some positive integer  $n$ .

16.  $|E(G)| = n - c$ .

17. There are many possible graphs satisfying these properties. If your graph is disconnected or has a cycle then it is not a tree.

18. (a) Does not exist.

(b) Does not exist.

(c) One such tree:



(d) Does not exist.

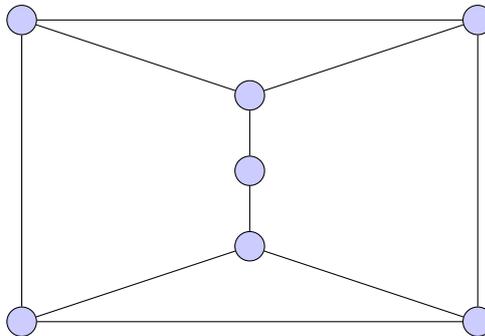
19. (a) There is no such spanning tree.

(b) The subgraph induced by vertices  $d, e, f, g$  is one such induced 4-cycle.

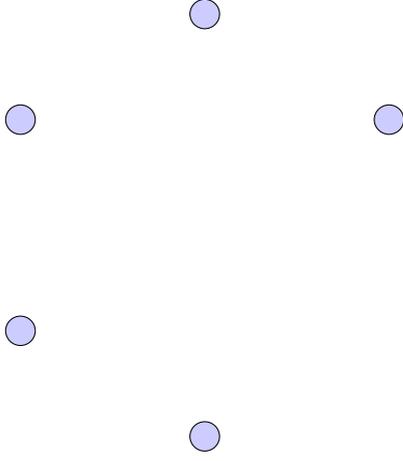
## 2.7 Planar Graphs

Solutions:

1. A graph is 'planar' if it can be drawn in the plane so that no edges cross.
2. Two graphs are homeomorphic if one is a 'subdivision' of the other.
3. (a) This graph is nonplanar as it is homeomorphic to  $K_{3,3}$ .  
(b) This graph is planar.



- (c) This graph is planar.



9. Hint: Draw  $P_2$ .
10. Hint: Use Euler's Theorem and Euler's Planar Graph Theorem.
11. Hint: Euler's Planar Graph Theorem.
12. (a) False.  
(b) True.  
(c) True.  
(d) True.
13. (a) Such a graph does exist and has exactly 8 regions.  
(b) Such a graph does exist and has exactly 9 edges.  
(c) There is no such planar graph.  
(d) There is no such planar graph.
14. Does not exist.
15. Hint: How many edges do these two regions share?

## 2.8 Colouring Graphs

### Solutions:

1. Assigning every vertex of a graph a colour such that no adjacent vertices have the same colour.
2. The graph is edge-less/empty.
3. (a)  $(K_n) = n$ .  
(b)  $(K_{m;n}) = 2$ .  
(c)  $(G) = 2$ :  
(d)  $(G) = 2$ :  
(e)  $(G) = 3$ .  
(f)  $(G) = 2$ :  
(g)  $(G) = 4$ .  
(h)  $(G) = 2$ .  
(i)  $(G) = 4$ .
4. False.
5. (a) False.  
(b) False.  
(c) True.  
(d) False.  
(e) False.  
(f) True.

(g) False.

(h) False.

(i) False.

(j) True.

(k) False.

6. (a)

7. (a) Every animal represents a vertex. Two vertices are adjacent (i.e. there is an edge) if these animals cannot live together peacefully. The vertices assigned the same colour represent the animals that can live in the same enclosure. The zoo is attempting to find the chromatic number of such a graph.
- (b) Let each course be a vertex, with two vertices adjacent if a student indicates that they would like to enroll in both courses. Vertices assigned the same colour represent courses that can run at the same time. Any colouring of this graph will give the department such a schedule, however

### 3 Counting: Fundamental Topics

#### 3.1 Basic Counting Principles

#### 3.2 The Rules of Sum and Product

Solutions:

1. (a) The sum rule for multiple events is for events that are independent (i.e. events/situations that cannot occur at the same time). If one event can occur in  $m$  ways, and the other event can occur in  $n$  ways, where both events are independent, then the two events together can occur in  $m+n$  ways.
- (b) The product rule for multiple events is for events that are happening in sequence of one another, thus are dependent. For example, if one event can occur in  $m$  ways and then another event follows and can occur in  $n$  ways, then the sequence of these two events can occur in  $m \times n$  ways.
2. The product and sum rule are used together when a set of sequences of events are occurring independently.
3. (a) Rule of product

36.

- (b) Rule of product

35 880 000.

- (c) Rule of product and rule of sum

25 309 440.

(d) Rule of product and rule of sum.

1 021 798 336.

(e) Rule of product and sum

6 718 464.

4. Rule of sum

58.

5. Rule of sum and product

231.

6. Rule of product

36.

7. Rule of product

22.

8. (a) Rule of product

5 9 10.

(b) Rule of product

$$5 \cdot 9!$$

(c) Rule of product

$$62 \cdot 784.$$

9. Rule of product

$$1 \cdot 536.$$

10. Rule of product

$$5 \cdot 9 \cdot 10.$$

11. Rule of product

$$456 \cdot 976.$$

12. Rule of product

(a)  $n^{m-1} \cdot 1.$

(b)  $n^{m-2} \cdot n \cdot 1.$

(c)  $3 \cdot n^{m-1}.$

(d)  $n \cdot n \cdot 1^{m-1}.$

(e)  $n^{m-1} \cdot n \cdot 1.$

13. Rule of product

$$3^5.$$

14. Rule of product

$$2^{25}.$$

15. Rule of product and sum

$$1\ 550.$$

16. Rule of product

$$60.$$

17. Rule of product

Lock 1 has a higher number of possible combinations and hence is the lock Jamie should purchase.

18. Rule of sum and product

$$18\ 278.$$

### 3.3 Permutations

#### Solutions:

1. A permutation is a linearly ordered arrangement of distinct objects.
2. If objects could be repeated, two identical arrangements would be counted as different.
3. (a) 120.  
(b) 60.  
(c)  $5^0$ .
4. 20.
5. (a) 5 040.  
(b) 1 440.
6. (a) 6 375 600.  
(b) 2 772.
7. 30 240.
8. 1 440.
9. 27 907 200.

10. 100.

11. (a) 3 315 312 000.

(b) 892 584 000.

(c) 127 512 000.

(d) 1 356 727 680.

12.  $m!(n + 1)!$

13. Applying the definition,

$$P(n; n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

14. The permutation function is one specific application of the product rule if you are trying to determine the number of possible arrangements of  $n$  distinct objects ( $k = n$ ). If this is not the specific case, count using the Rule of Product.

15. Solve your own problem to verify this.

16. Hint: Use the definition of the permutation function.

17. 5 184.

18. 34 650.



### 3.4 Combinations and the Binomial Theorem

#### Solutions:

1. The main difference between a permutation and a combination is that for a

$$(b) \binom{10}{5} \binom{2}{1}.$$

$$7. \binom{10}{3} \binom{8}{1} = \binom{9}{2} + \binom{9}{2} + \binom{8}{3}.$$

$$8. \binom{15}{7} \binom{12}{5} = \binom{2}{1} \binom{13}{6} + \binom{13}{7} + \binom{12}{4}.$$

$$9. (a) \binom{20}{8}.$$

$$(b) \binom{10}{4} \binom{10}{4}.$$

$$(c) \binom{20}{8} \binom{18}{6} = \binom{18}{8} + \binom{2}{1} \binom{18}{7}.$$

$$(d) \binom{20}{8} \binom{8}{1} = \binom{20}{1} \binom{19}{7}.$$

10. Algebraic:

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$



equivalent.

13. (a)  $\sum_{k=0}^n x^k y^{n-k} \binom{n}{k}$ .

(b)  $\sum_{k=0}^6 (3)^k (x)^{6-k} \binom{6}{k} = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$ .

(c)  $\sum_{k=0}^7 (2x)^k (3y)^{7-k} \binom{7}{k} = 128x^7 + 1344x^6y + 6048x^5y^2 + 18$

- (b)  $2^{3n}$ .
- (c)  $(-1)^n$ .
- (d)  $6 \cdot 2^n$ .
- (e) 0.

19. When evaluating a polynomial with more than two terms to some integer power the multinomial theorem is used to determine the coefficients of the terms. The binomial theorem is a version of the multinomial theorem that can be used for binomials.

20. (a) 12.  
(b) 816 480.  
(c) There is no term with  $xyz$ .

### 3.5 Combinations with Repetitions

Solutions:

1. i) The number of ways identical elements can be distributed into ~~to~~ distinct containers.
- ii) The number of non-negative integer solutions to:

$$x_1 + x_2 + \dots + x_n = r:$$

2. We are selecting ~~ing~~ elements, with possible repetition, from a set of ~~of~~ distinct objects.

3. 330.

4. 41 120 525.

5. (a)  $\binom{40}{12}$ .
- (b)  $\binom{51}{12}$ .

6. (a)  $\binom{20+m-1}{20}$ .
- (b)  $m^{20}$ .

7. 1 680.

8. There are many possible correct answers, one example is, in how many can you distribute ten treats between three dogs?

9. 42 504.

10. 56.

11. 2 925.

12. 26 334.

13. 126.

14.  $n = 7$ :

15.  $n = 7$ .

16. Algebraically : Expanding using the definition of the combination function, we know,

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!},$$

and that,

$$\binom{n+r-1}{n-1} = \frac{(n+r-1)!}{(n-1)!r!}.$$

It is easy to see that these expansions are equal since multiplication is commutative, so we're done.

Combinatorial proof : We can look at this problem as placing balls into boxes.

If we wish to do this, we can line up  $r$  balls and place  $n-1$  dividers between

them. The balls between either the beginning/end and a divider or two dividers represents the number of balls in a box. Thus, there are  $n+r$  total positions, where each position is either filled with a ball or a divider.

From these positions we can either choose where to first place the dividers and then have the balls in the remaining positions,  $\binom{n+r-1}{r}$ , or we can choose where to place the balls first and have the dividers in the remaining positions,  $\binom{n+r-1}{n}$ .

The two options are equivalent and are equal to the right and left sides of the equation, respectively.

17. (a)  $P(n; r)$   
 (b)  $C(n; r)$   
 (c)  $n^r$   
 (d)  $C(n+r-1; r) = C(n+r-1; n-1)$

18. 136.

19.

(c)

$$x_1 + x_2 + x_3 + x_4 = 12;$$

where  $x_i \geq 2$  for  $i = 1, 2, 3, 4$ . Each  $x_i$  represents the number of apples of each variety that have been selected.

(d)

$$x_1 + x_2 + x_3 + x_4 = 15;$$

where  $x_i \geq 0$  for  $i = 1, 2, 3, 4$  with  $x_1 = x_2$ . Each  $x_i$  represents the amount of markers in each box.

20.  $\binom{n+m+x-1}{x} = \binom{n}{m}$ .

21.  $\binom{m+s-1}{s} = \binom{n+r-s-1}{r-s}$ .

### 3.6 The Pigeonhole Principle

#### Solutions:

1. If there are  $n$  pigeons that are flying into  $k$  pigeonholes where  $n > k$ , then there must be at least one box with at least two pigeons.
2. A function from one finite set to a smaller finite set cannot be one-to-one. There

The pigeons are the different integers and the pigeonholes represent the possible remainders upon division by 28.

- (c) You must take out at least 10 shoes before you are guaranteed to obtain a pair. Any less and it is possible that each shoe is from a different pair.

The pigeons are the shoes and the pigeonholes are the pairs of shoes.

- (d) To determine this, we must first count the number of distinct three letter words. For each position in the word there are 26 possible letters. Therefore, in total there are  $26^3 = 17576$  distinct three letter words. This means that it is possible, but not definitive, that all words on this list are distinct as there are more three-letter words possible than there are words on the list.

The pigeons are the number of three-letter words and the pigeonholes are the words on the list.

6. Hint: Prove the statement about subsets of size 6, the result will follow for all larger subsets.

7. (a) Any even positive integer can be written as  $2^k y$  (essentially just factoring out the two's), where  $k \in \mathbb{N}$ , and  $y$  is odd. There are exactly 1000 odd numbers in  $A = \{1, \dots, 2000\}$ .

Let us define a pigeonhole for each odd integer  $y \in A$  as:

$$PH_y = \{x \in A : x = 2^k y; \text{ where } y \text{ is odd and } k \in \mathbb{Q}\}.$$

This gives us our 1000 pigeonholes.

Select any 1001 numbers from  $A$ , these are the pigeons. Then, by the pigeonhole principle, there exists a pigeonhole  $PH_y$ , that contains two

selected numbers  $a$  and  $b$ . Say  $a = 2^k y$  and  $b = 2^p y$  for some distinct, nonnegative integers  $k$  and  $p$ . If  $k > p$ , then  $b$  divides  $a$  and if  $k < p$ , then  $a$  divides  $b$ .

(b) We can partition the numbers into sets of size two, where the second

Note Recognizing that you need to do this is the heart of the proof. We add 23 because we are trying to prove that there are 23 consecutive days where scholarship applications were sent out, which allows us to conclude this by the PHP.

Now we have 84 distinct numbers,  $x_1, x_2, \dots, x_{42}, x_1 + 23, x_2 + 23, \dots, x_{42} + 23$ . Let these numbers represent out 'pigeons'. These 84 numbers must all lie between 1 and 83, where the range of integers from 1 to 83 represent out pigeonholes. Thus by the PHP there exists  $x_j + 23$  for some  $j \in \{1, \dots, 42\}$ . This means that from the beginning of day 1 to the end of day 23, Brynn applied for 23 scholarships.

10. It suffices to prove the result for subsets of exactly size three, since that will imply the result for subsets of size larger than three.

The only way for the sum of two integers to be even is if both of the integers have the same parity, that is both are even or both are odd. Any given integer can be classified as either even or odd, hence any subset of 3 integers will contain at least two with the same parity by the PHP. Thus, there are two integers in the subset with an even sum.

11. There are 12 pigeonholes (computers) and 42 pigeons. In this problem there is a restriction that no pigeonhole can hold more than 6 pigeons.

We wish to show that there are five computers which are used by three or more people.

Let us assume for a contradiction that this is not true. This would mean that 8 computers are used by at most 2 people. This would mean that these 8 computers are used by at most 16 people all together.

There are 42 people who use a computer at the library and so that means the

remaining 26 people use 4 computers.

This means that there are 26 pigeons and 5 pigeon holes where the maximum capacity of each pigeonhole is 6. This however gives that the maximum capacity for the remaining computers is 24, which is a contradiction.

So, at least 5 computers are to be used by three or more people.

12. Let us assume that if one person speaks to another, the person will respond. That is, assume speaking to someone is a reflexive relation.

If there are  $n$  people at the party, each person can speak to between 0 and  $n - 1$  people, as no person can speak to themselves and speaking to someone is reflexive.

If a person spoke to  $n - 1$  people, then it is impossible for any person to have spoken to 0 people. In this case every person spoke to between  $n - 1$  and  $n - 1$  people. That means there are  $n - 1$  potential number of people a person could have spoken to.

If a person at the party spoke to 0 people, then it is impossible for someone to have spoken to everyone. In this case every person will have spoken to between 0 and  $n - 2$  people. That means there are  $n - 1$  potential number of people a person could have spoken to.

In both of the above cases, there are  $n - 1$  potential amounts of people a person could have spoken to but  $n$  people. Thus, by the PHP two people will have spoken to the same amount of people at the party.

13. For 12 to divide the difference of two numbers, they must have the same remainder upon division by 12. Observe that  $12n + (12j + m) = 12(k + j) + m$ ,



## 4 Inclusion and Exclusion

### 4.1 The Principle of Inclusion-Exclusion

Solutions:

1. The Principle of Inclusion-Exclusion is a counting method that ensures every possible event is counted, while also taking into account events that can co-occur. This method is a way of ensuring events are not counted twice or "double counted".
2. False.
3. (a) 16.  
(b) 27.  
(c) 20.
4. 300.
5. (a) 46.  
(b) 35.
6. (a) There is no correct answer - the question is unanswerable because the situation described is impossible.  
(b) 0.  
(c) 135.

7.  $11! \sum_{i=1}^6 \binom{6}{i} (-1)^i (11-i)!$



## 4.2 Derangements: Nothing in its Right Place

Solutions:

9. (a)  $D(12)$ .  
 (b)  $D(6)$ .  
 (c)  $\sum_{k=0}^{12} (-1)^k [(12-k)!] \binom{12}{k}$ .

Hint: Use PIE instead of derangements.

10. We will count the number of permutations of the numbers  $1, 2, \dots, n$ , which is certainly  $n!$ . Alternatively, for every possible permutation we can consider how there are  $k$  elements that have been deranged, and hence  $n-k$  elements in their original positions for  $0 \leq k \leq n$ . The  $n-k$  fixed elements can be selected in  $\binom{n}{n-k} = \binom{n}{k}$  ways, with  $D(k)$  ways that the remaining elements can be deranged. We sum these cases from  $k=0$  to  $k=n$  to account for all possible permutations, and the proof is complete since we've counted the same situation in two different ways.

### 4.3 Onto Functions and Stirling Numbers of the Second Kind

Solutions:

Note: There are two equivalent formulas for Stirling numbers:

$$S(m; n) = \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m = \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m$$

These formulas are equivalent as:

$$\begin{aligned} \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m &= \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m + (-1)^n \binom{n}{n} (n-n)^m \\ &= \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m + 0 \\ &= \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m \end{aligned}$$

Thus, you may use either in your solutions.

1. An onto function is a function  $f: A \rightarrow B$  where for all  $b \in B$ , there exists some  $a \in A$  such that  $f(a) = b$ .
2. The number of ways to distribute  $n$  different objects into  $m$  distinct containers where no container is left empty is  $S(n, m) \cdot m!$ .
3. There are many possible examples, one example is the function where  $x \in \mathbb{Z}$

5. No such surjective function.

$$6. \sum_{k=0}^8 \binom{9}{k} (9-k)^{13}.$$

7. 2.

$$8. \sum_{k=0}^6 \binom{7}{k} (7-k)^{27}.$$

9. 2 100.

Hint: Two cases.

10. A Stirling number of the second kind, denoted  $S(m; n)$ , is the number of ways to distribute  $m$  distinct objects into  $n$  identical containers with no container left empty.

The formula is:  $S(m; n) = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k (n-k)^m$ . This is the formula for counting the number of onto functions from a set of size  $m$  to a set of size  $n$  divided by  $n!$ . The division by  $n!$  is done to account for the identical "containers".

11.  $j! S(k; j)$ .

12. (a) 5 103 000.

(b) 1 020 600.

Hint: Cases depending on if any other studied animals go into the first bin.

(c) 86 472.

Hint: Six cases.

13. We recall from number theory that any factor of 55 335 will be the product of some subset of the factors of 55 335. For example 5 and 13 17 are both factors of 55 335.

(a)  $S(5; 2) = 15$ .

(b)  $\sum_{n=2}^5 S(5; n) = 51$ .

Hint: Four cases.

14. Hint: Expand the sums.

## 5 Generating Functions

### 5.1 Introductory Examples

### 5.2 Definition and Examples: Calculating Techniques

Solutions:

$$1. (a) \quad 1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n+1} + \dots = \sum_{k=0}^{\infty} \binom{1}{k} x^k = \frac{1}{(1-x)^2}$$

$$(b) \quad 5 + 4x + 3x^2 = \sum_{k=0}^2 \binom{2}{k} \binom{5}{1-k} x^k:$$

$$(c) \quad 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots = \sum_{k=0}^{\infty} \binom{1}{k} (-x)^k = \frac{1}{1+x}:$$

$$(d) \quad \binom{10}{10} + \binom{11}{10} x + \binom{12}{10} x^2 + \dots = \sum_{k=0}^{\infty} \binom{1}{k} x^k \binom{10+k}{10} = \frac{1}{(1-x)^{11}}$$

$$(e) \quad \binom{10}{10} + \binom{11}{10} x + \binom{12}{10} x^2 + \binom{13}{10} x^3 + \dots = \sum_{k=0}^{\infty} \binom{1}{k} (-x)^k \binom{10+k}{10} = \frac{1}{(1+x)^{11}}$$

$$(f) \quad 1 + x^2 + x^4 + \dots = \sum_{k=0}^{\infty} \binom{1}{k} x^{2k} = \frac{1}{1-x^2}$$

$$(g) \quad 1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 = \sum_{k=0}^5 \binom{5}{k} (-2x)^k = \frac{1}{1-(2x)^6} = \frac{1-64x^6}{1+2x}$$

2. (a) 0000:::

(b) 01;000::

(c) 43; 1055

(d) 64144; 10827.

(e) 03;3;3;3:::

(f) 1;6;27;108:::

3. (a) 1.

(b) 24.

(c) 1.

(d) 57 915.

4. 3 246.

5. (a) 120.

(b) 48.

(c) 21.

6.

(b)

$$g(x) = \frac{(1-x^6)(1-x^3)}{(1-x)^2(1-x^2)^2}$$

(c)

$$g(x) = x^{10} \frac{1}{(1-x)^4}$$

(d)

$$g(x) = \frac{1}{(1-x^2)(1-x^3)^2(1-x)}$$

12. The coefficient of  $x^{50}$  in,

$$g(x) = \frac{1}{1-x} \frac{1}{1-x^5} \frac{1}{1-x^{10}} \frac{1}{1-x^{25}}$$

13. (a) 8.

(b) 90.

14. (a)

$$g(x) = (1+x+x^2+x^3+\dots)^3$$

(b)

$$g(x) = x^3(1+x+x^2+x^3+\dots)^3$$

15. 6.

### 5.3 Partitions of Integers

Solutions:

1. A partition on a positive integer is a collection of unordered positive integers that all

$$5 = 1 + 1 + 1 + 1 + 1$$

$$5 = 1 + 1 + 1 + 2$$

$$5 = 1 + 2 + 2$$

$$5 = 1 + 1 + 3$$

$$5 = 1 + 4$$

$$5 = 2 + 3$$

$$5 = 5$$

6.

$$g(x) = (1 + x + x^2 + \dots)^5$$

7. (a)

$$g(x) = x^k \prod_{i=1}^k \frac{1}{(1 - x^i)}$$

8. Hint: Find the two generating functions and explain why they're equal.
9. Hint: Compare the relevant generating functions.
10. Hint: Find a one-to-one correspondence between any partitioning of  $n$  into  $a$  parts and any partitioning of  $n$  into  $b$  parts.
11. Hint: Find a one-to-one correspondence between the two sets.

## 6 Recurrence Relations

### 6.1 First-Order Linear Recurrence Relations

Solutions:

1. A recurrence relation is an expression for a function  $f(n)$  that is defined in terms of previous terms, such as  $f(n) = 2f(n-1)$ , with one or more initial values for  $f(k)$  stated.
2. Solving a recurrence relation means determining a function, whose domain is the set of non-negative integers, that describes the recurrence relation for all  $n \geq 0$  without solving for previous terms.

3.  $a_6 = 262$

4.  $a_n = 5 \cdot (2)^n$ , for  $n \geq 0$ .

5.  $a_n = 909 \cdot \left(\frac{1}{3}\right)^n$ , for  $n \geq 0$

6.  $a_n = k \cdot \left(-\frac{6}{5}\right)^n$ .

7.  $a_n = \frac{1296}{2401} \left(\frac{7}{2}\right)^n$ .

8. \$16625.

9.  $b_n = 25 \cdot 3^n$ , for  $n \geq 0$ .

Hint: Make the substitution  $b_n = a_n^2$ .

10. (a)  $a_0 = 0$ , and  $a_{n+1} = a_n + 2n$ , for  $n \geq 1$ .

(b)  $a_0 = 7$ , and  $a_{n+1} = \frac{2a_n}{5}$ , for  $n \geq 1$ .

11.  $d = \frac{2}{7}$ .

12.  $5(3^{36})$ :

13. (a)  $a_n = (-5)^n$ ,  $n \geq 0$ .

(b)  $a_n = (4)^{n-1}$ ,  $n \geq 1$ :



$$a_n = \left( 1 - \frac{p\sqrt{412}}{2\sqrt{5423}} \right) \left( \frac{74 + p\sqrt{5423}}{53} \right)^n + \left( -\frac{p\sqrt{413}}{2\sqrt{5423}} - 1 \right) \left( \frac{74 + p\sqrt{5423}}{53} \right)^n \text{ for } n \geq 0$$

5.  $a_n = 5a_{n-1} - 11a_{n-2}$  for  $n \geq 2$ . The initial conditions are  $a_0 = a$ ;  $a_1 = b$  for any  $a, b \in \mathbb{R}$ .

6.  $a_n = a_{n-1} + a_{n-2}$

(b)  $u_n = 2u_{n-2} + u_{n-1}$ , where  $u_0 = 1$ .